Module -I

Oscillations and Waves

1. Oscillatory motion:

The to & fro motion of a particle about an equilibrium position is called oscillatory motion or oscillation. The particle exhibiting such type of motion is called oscillator or if it is a system it is called oscillatory system.

Example: Simple pendulum, pendulum of a wall clock, balance wheel of a watch. String of musical instruments.

Characteristics:

1. In oscillatory motion there is an equilibrium position about which the particle oscillates.
2. A restoring force always acts which tries to bring the particle back to the mean position.
3. Depending upon the external force applied, an oscillatory system can be classified into 3 categories, i.e.,

a. Free oscillatory system $\rightarrow$	Not acted upon by any external force.
b. Damped oscillatory system $\rightarrow$	Acted upon by external resistive force.
c. Forced oscillatory system $\rightarrow$	Acted upon by external periodic force along with the resistive force such that the amplitude of oscillation increases.

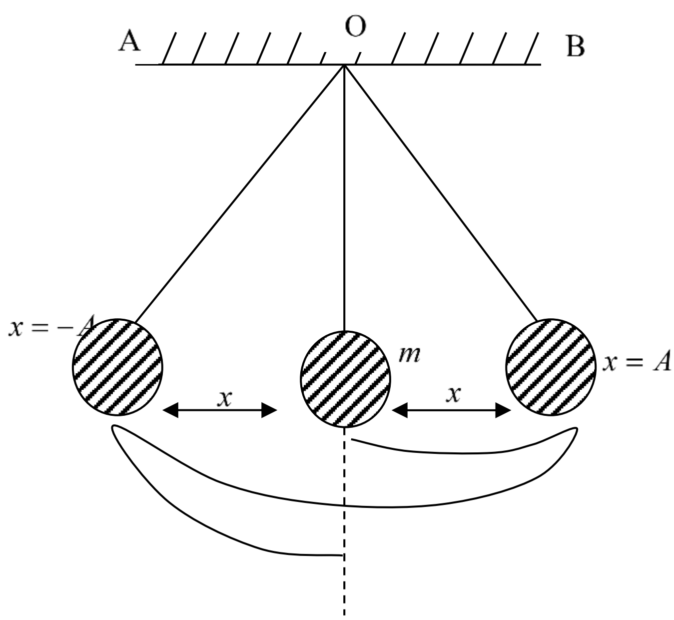
2. Simple Harmonic Motion (SHM) :

It is the simplest type of oscillatory motion in which the particle oscillates about an equilibrium position. Since the motion is expressed in terms of trigonometric harmonic functions so it is called simple harmonic motion (SHM).

Or

It is that type of motion in which the particle oscillates about the mean position and a restoring force acts in a direction apposite to the displacement & tries to bring the particle back to the mean position.

The equation of motion for a particle executing Simple Harmonic Motion can be obtained by equating the forces acting on it.



Equation of Motion:

Consider a particle of mass 'm' executing SHM with displacement 'x'. The forces acting are,

(i) The force responsible for harmonic motion and is given by,

$$F_1 = ma = m\frac{d^2x}{dt^2} \quad ---- (1)$$

(ii) The restoring force proportional to the displacement and takes place in a direction opposite to it,

$$F_{res} \propto -x$$

$$\Rightarrow F_{res} = -kx \qquad ------- (2)$$

where, k = proportionality constant called spring constant or force constant.

Under equilibrium condition,

$$F_1 = F_{res}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad ------ (3)$$

Where, $\omega = \sqrt{k/m}$ = angular frequency.

Equation (3) represents the equation of motion for a particle executing simple harmonic motion. It is a second order differential equation & its solution can be obtained as follows,

Let the solution be,

$$x = Ae^{\lambda t} \quad \Rightarrow \frac{dx}{dt} = A\lambda e^{\lambda t} \quad \Rightarrow \frac{d^2x}{dt^2} = A\lambda^2 e^{\lambda t}$$

Substituting the above values, equation (3) can be written as,

$$A\lambda^2 e^{\lambda t} + \omega^2 Ae^{\lambda t} = 0$$

$$\Rightarrow Ae^{\lambda t}\left(\lambda^2 + \omega^2\right) = 0$$

$$\Rightarrow \left(\lambda^2 + \omega^2\right) = 0 \qquad \text{as}, Ae^{\lambda t} \neq 0 \text{ being the solution}$$

$$\Rightarrow \lambda^2 = -\omega^2 \quad \Rightarrow \lambda = \sqrt{\left(-\omega^2\right)} \quad \Rightarrow \lambda = \pm i\omega \quad i = \sqrt{(-1)} = \text{Imaginary number}$$

So, the solution can be written as,

$$x = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

$$\Rightarrow x = A_1\left(\cos\omega t + i\sin\omega t\right) + A_2\left(\cos\omega t - i\sin\omega t\right)$$

$$\Rightarrow x = \left(A_1 + A_2\right)\cos\omega t + i\left(A_1 - A_2\right)\sin\omega t$$

By taking, $\left(A_1 + A_2\right) = A\sin\theta$ and $i\left(A_1 - A_2\right) = A\cos\theta$, we can write,

$$x = A\sin\theta\cos\omega t + A\cos\theta\sin\omega t$$

$$\Rightarrow x = A\sin\left(\omega t + \theta\right) \qquad \text{----------------(4)}$$

θ = constant that determines the phase of the particle.

If there is no phase difference, θ = 0, then, equation(4) can be written as,

$$x = A\ \mathrm{Sin}\omega t \qquad \text{----------- (5)}$$

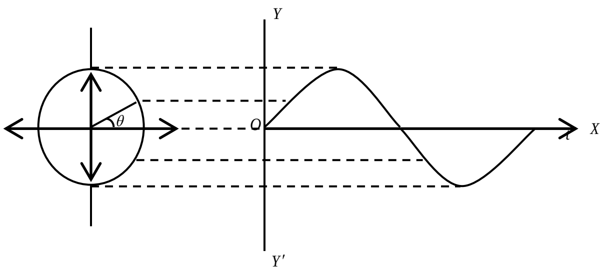
Equation (5) represents the displacement of a particle executing SHM.

With different types of substitution, the solution can also be written in different form which physically represents the same SHM.

They are, $x = A\sin\left(\omega t - \theta\right)$, $x = A\cos\left(\omega t + \theta\right)$, $x = A\cos\left(\omega t - \theta\right)$

Characteristics of SHM:

a. SHM can be represented as the projection of uniform circular motion.



b. **Displacement (x):** Displacement of a particle is defined as the distance traveled by the Particle on either side of the mean position.

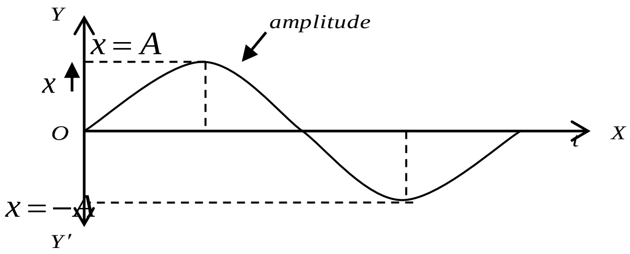
Mathematically, x = Asinωt (considering no phase difference)

when, ωt = 0, $\Rightarrow$ x = 0 ---------------------(a)

For, ωt = $\pi/2$, $\Rightarrow$ x = A ---------------------(b)

x = A is the Maximum Displacement & is called the **amplitude.**

So, amplitude is defined as the maximum displacement made by the oscillator on either side of the mean position.



c. **<u>Velocity</u> (v):** It is defined as the time rate of change of displacement.

We know that, $x = A\sin\omega t$

$$\Rightarrow \quad \frac{dx}{dt} = v = A\omega\cos\omega t$$

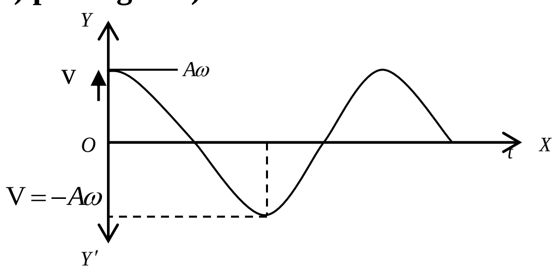
$$\Rightarrow \quad v = \omega A(1 - \sin^2\omega t)^{1/2} \Rightarrow v = \omega(A^2 - A^2\sin^2\omega t)^{1/2}$$

$$\Rightarrow \quad v = \omega(A^2 - x^2)^{1/2} \quad \text{-----------} \quad (6)$$

<u>Case– I</u> **When, $x = 0$ (equilibrium position)**

$\Rightarrow$ v = Aω = maximum value **(From equation (6) putting x=0)**

$\Rightarrow$ Velocity is maximum at mean position and is called velocity amplitude.



<u>Case – II</u> **When, x =A (extreme position)**

$\Rightarrow$ v =0 = minimum value. **(From equation (6) putting x=A)**

$\Rightarrow$ Velocity is minimum at extreme position of displacement

d. **Acceleration (a):** The rate of change of velocity of a particle executing SHM is called acceleration.

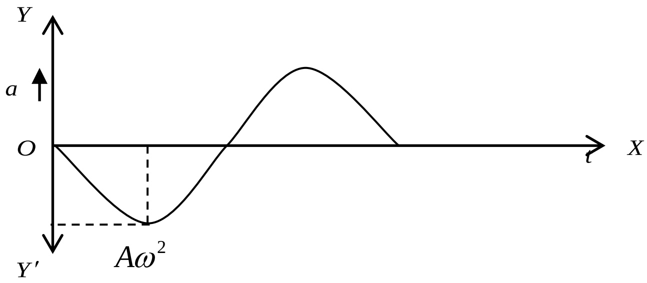
We know that, $$a = \frac{dv}{dt} = \frac{d(A\omega\cos\omega t)}{dt}$$

$$\Rightarrow \quad a = -A\omega^2\sin\omega t = -\omega^2 x \quad \text{-----------} \quad (7)$$

<u>Case– I</u> **When, x = 0 (equilibrium position)**

$\Rightarrow$ a = 0 = minimum value

$\Rightarrow$ Acceleration is minimum at mean position.



<u>Case – II</u>

When, x =A (extreme position)

$\Rightarrow$ a = - ω^2A$\rightarrow$ maximum value

$\Rightarrow$ Acceleration is maximum at extreme position and is called acceleration amplitude

e. **Time Period (T):** It is defined as the time taken by the particle to complete one oscillation.

We know that SHM is the projection of uniform circular motion. Therefore, $\omega = \frac{\theta}{t}$

When, $t = T$, $\theta = 2\pi$ so that, $\omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow T = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{m/k}$ -----------(8)

In terms of displacement and acceleration,

$T = 2\pi\sqrt{x/a}$ -----------------------------(9) $(\because a = \omega^2 x, \omega = \sqrt{a/x})$

f. Frequency (f): It is defined as the total number of oscillations per seconds & is denoted by 'f'.

In 'T' seconds, number of oscillations is 1,

In 1 second, number of oscillations is $\frac{1}{T}$

So, $f = \frac{1}{T} \Rightarrow f = \frac{1}{2\pi\sqrt{m/k}} \Rightarrow f = \frac{1}{2\pi}\sqrt{k/m}$ ----------(10)

In terms of 'x' and 'a', $f = \frac{1}{T} \Rightarrow f = \frac{1}{2\pi\sqrt{x/a}} \Rightarrow f = \frac{1}{2\pi}\sqrt{a/x}$ ---------------(11)

g. Phase (ϕ): The physical quantity that describes the position as well as the direction of motion of the particle executing SHM is called phase of the particle. The phase also defines the initial position of the oscillator

The phase difference can be represented as $x = A \sin(\omega t \pm \varphi)$.

h. Energy (E): The energy of a particle executing SHM is the sum of Kinetic Energy and Potential Energy of the particle and is constant.

The Total Energy is, E = K.E. + P. E. = $E_k + E_p$ = Constant

(i) Kinetic Energy: It is the amount of energy possessed by the particle by virtue of its motion.

Mathematically,

$$E_K = \frac{1}{2} mv^2 = \frac{1}{2} mA^2\omega^2 \cos^2 \omega t = \frac{1}{2} m\omega^2(A^2 - x^2)$$

(ii) Potential Energy: It is the amount of energy possessed by the particle by virtue of its position. It can also be defined as the net amount of work done by the system or on a system to get it displaced from one point to the other.

Let the particle be displaced a small distance dx from the mean position.

$$dW = \vec{F} \bullet d\vec{x} = F dx \cos 180^0 = - F dx$$

$(\because \vec{F}$ & $d\vec{x}$ are in opposite directions to each other)

Now, the total work done in moving the particle from a displacement of x to equilibrium position can be written as,

$$W = \int dW = \int_x^0 (-)F dx$$

$$= \int_x^0 -kx dx = -k\int_x^0 x dx = -k\left[\frac{x^2}{2}\right]_x^0$$

$$= \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

$$(\because \omega = \sqrt{k/m} \Rightarrow k = m\omega^2)$$

So, $E_p = \frac{1}{2}m\omega^2x^2$

Now,

Total energy is, $E = \mathbf{E_k} + \mathbf{E_p} = \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2$

⇒ For a particular oscillation since m, ω and A are constants, so the total energy of a particle executing S.H.M. is constant.

<u>Case-I</u> **When x = 0 (mean position)**

$E_K = \frac{1}{2}m\omega^2A^2$(maximum value) and $E_P = 0$ (minimum value).

⇒ At mean position E_P is minimum and E_K is maximum, but the Total Energy,

$E = \frac{1}{2}m\omega^2A^2$ is constant

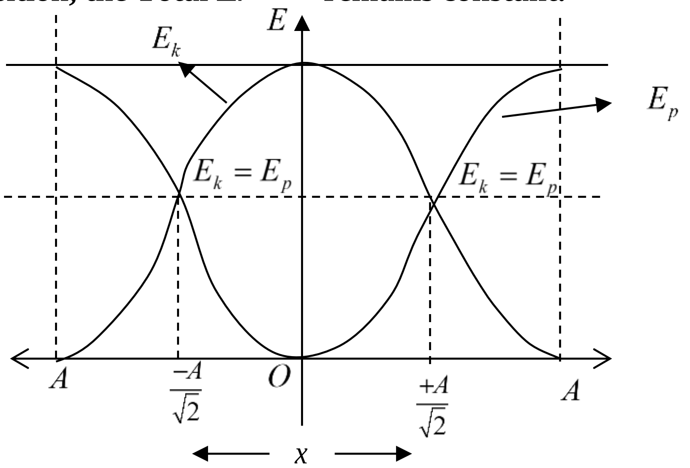
<u>Case-II</u> **When x = A (extreme position)**

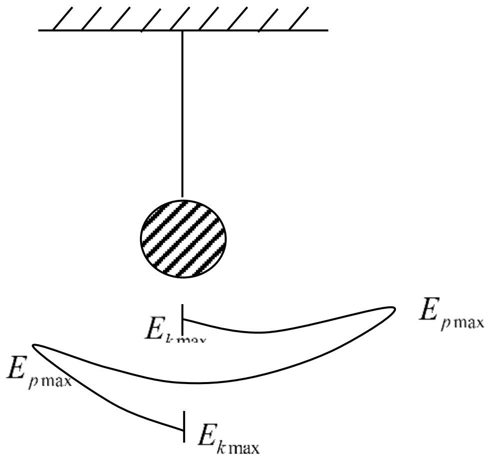
$E_K = 0$ (minimum value) and $E_P = \frac{1}{2}m\omega^2A^2$(maximum value).

⇒ At extreme position E_P is maximum and E_K is minimum, but the Total Energy,

$E = \frac{1}{2}m\omega^2A^2$ is constant

So, it can be concluded that whether the oscillator is at mean position or at extreme position, the Total Energy remains constant.





Q1. What is the value of displacement at which P.E is equal K.E.

ANS.: Given condition is that, P.E = K.E

$\Rightarrow \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2(A^2 - x^2) \Rightarrow A^2 = 2x^2 \Rightarrow x = \pm\frac{A}{\sqrt{2}}$.

When the amplitude falls to $\frac{1}{\sqrt{2}}$ of its maximum value, the Potential Energy becomes equal to Kinetic Energy

Examples of S.H.M.:

Mass suspended by means of a sting.	Atoms and ions in a solid.
Pendulum of wall clock.	Balance wheel of watch.
Simple pendulum.	Torsion pendulum.

3. DAMPED HARMONIC MOTION(DHM) :

It is that type of harmonic motion in which in addition to the restoring force, an external resistive force such as frictional force or viscous force acts which opposes the motion and as a result the oscillation gets damped and hence the name Damped Harmonic Motion (DHM) or Damped Oscillatory Motion.

Equation of Motion

Consider a particle of mass 'm' executing DHM with displacement 'x'. The forces acting are,

(i) Force responsible for harmonic motion.

$$F_1 = m\frac{d^2x}{dt^2} \quad \text{--------------} \quad \textbf{(1)}$$

(ii) The restoring force proportional to the displacement and takes place in a direction opposite to it.

$$F_2 = -kx \quad \text{---------------} \quad \textbf{(2)}$$

k = proportionality constant known as spring constant.

(iii) External resistive force proportional to the velocity and takes place in a direction opposite to it.

$$F_3 \propto -\frac{dx}{dt}$$

$$\Rightarrow F_3 = -R\frac{dx}{dt} \quad \text{-----------------} \quad \textbf{(3)}$$

R = Proportionality constant and is called frictional force per unit velocity. It is also called as damping constant.

Under equilibrium Condition,

$$F_1 = F_2 + F_3$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx - R\frac{dx}{dt}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{R}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = \frac{F}{m}\cos\omega_1 t \quad \text{----------} \quad \textbf{(4)}$$

Equation (4) represents the equation of motion for a particle executing damped oscillating motion.

Where, $\beta = \frac{R}{2m} =$ Damping co-efficient and $\omega = \sqrt{k/m}$ =the restoring force.

It is a second order differential equation whose solution can be obtained as follows:

Let the solution be,

$$x = Ae^{\lambda t} \quad \Rightarrow \frac{dx}{dt} = A\lambda e^{\lambda t} \quad \Rightarrow \frac{d^2x}{dt^2} = A\lambda^2 e^{\lambda t}$$

Now equation (4) can be written as,

$$\lambda^2 e^{\lambda t} + 2\lambda\beta e^{\lambda t} + \omega^2 e^{\lambda t} = 0$$

$$\Rightarrow (\lambda^2 + 2\beta\lambda + \omega^2)e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 + 2\beta\lambda + \omega^2 = 0 \quad \text{As} \quad e^{\lambda t} \neq 0 \text{ being the solution.}$$

$$\Rightarrow \lambda = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2} \quad \Rightarrow \quad \lambda = -\beta \pm (\beta^2 - \omega^2)^{1/2}$$

So, the solution can be written as,

$$x = A_1 e^{\left[-\beta + (\beta^2 - \omega^2)^{1/2}\right]t} + A_2 e^{\left[-\beta - (\beta^2 - \omega^2)^{1/2}\right]t}$$

$$\Rightarrow x = e^{-\beta t}\left[A_1 e^{\left\{(\beta^2 - \omega^2)^{1/2}\right\}t} + A_2 e^{-\left\{(\beta^2 - \omega^2)^{1/2}\right\}t}\right] \quad \text{-------- } \mathbf{(5)}$$

Where, A_1 & A_2 are constants whose values can be obtained by using proper boundary conditions.

Equation **(5)** represents the general solution of the differential equation for a particle executing damped harmonic motion.

The presence of $e^{-\beta t}$ term implies that the amplitude of oscillation decays exponentially. But the presence of ω term implies that there is possibility of oscillation. Depending upon the relationship between β & ω, we have three types of damped oscillatory motion.

(i) $\beta > \omega$ $\Rightarrow$ overdamped motion.

(ii) $\beta \cong \omega$ $\Rightarrow$ critically damped motion

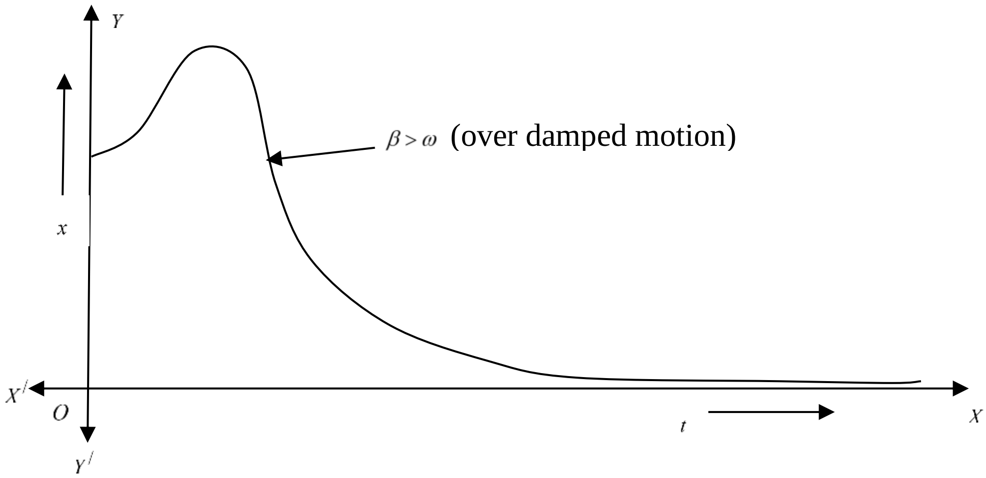
(iii) $\beta < \omega$ $\Rightarrow$ underdamped Harmonic motion

<u>Case-I:</u> **$\beta > \omega$ $\Rightarrow$ Overdamped Motion**

$\Rightarrow$ Frictional force > restoring force $\Rightarrow$ $(\beta^2 - \omega^2)$ = positive & real = r (say)

Therefore**,** Equation **(5)** can be written as,

$$y = e^{-\beta t}[A_1 e^{rt} + A_2 e^{-rt}] \Rightarrow y = A_1 e^{-(\beta - r)t} + A_2 e^{-(\beta + r)t} \quad \text{-------- } \mathbf{(6)}$$



Example : Particle allowed to oscillate in a high viscous fluid.

<u>Case-II:</u> **β ≅ ω ⇒ Critically Damped Motion**

⇒ Frictional force = restoring force ⇒ $(\beta^2 - \omega^2)^{1/2} = h$ (let a very small quantity)

So, equation (5) can be written as,

$$x = e^{-\beta t}[A_1 e^{ht} + A_2 e^{-ht}]$$

Taking Taylor series expansion of e^{ht} and e^{-ht} we can write

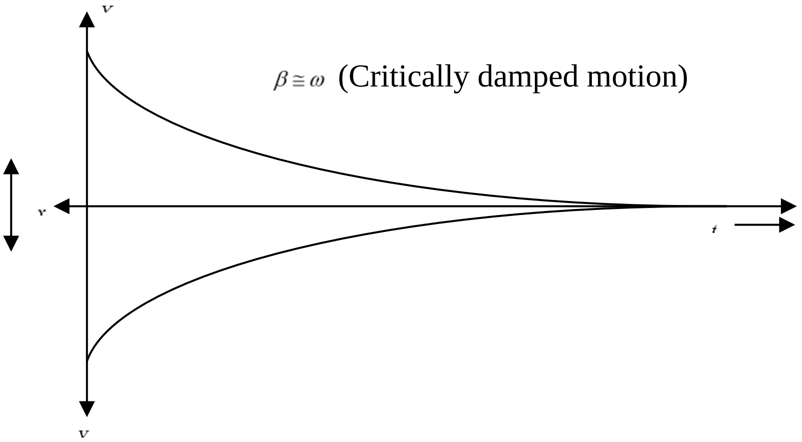
$$\Rightarrow x = e^{-\beta t}\left[A_1\left(1 + ht + \frac{(ht)^2}{2!} + \cdots\right) + A_2\left(1 - ht + \frac{(ht)^2}{2!} - \cdots\right)\right]$$

$$\Rightarrow x = e^{-\beta t}[A_1 + A_1 ht + A_2 - A_2 ht]$$ (since h is very small, so higher powers of h can be neglected)

$$\Rightarrow x = e^{-\beta t}[(A_1 + A_2) + (A_1 - A_2)ht] \qquad \Rightarrow x = e^{-\beta t}[A + Bt] \quad \text{--------} \qquad \textbf{(7)}$$

Where, $A = (A_1 + A_2)$ and $B = (A_1 - A_2)\, h$.

From the above expression it can be concluded that due to the appearance of time t in the coefficient, the oscillatior comes to equilibrium position in a short period of time as compared to overdamped motion. However, the oscillator does not oscillate due to $e^{-\beta t}$ term.



Examples:

(a) The suspension springs of automobiles. The springs are adjusted for critically damped motion such that the equilibrium position is attained within the very shortest possible time such that the person sitting in the automobile will feel less jerk.

(b) Galvanometer adjusted for critical damping. (Dead Beat Galvanometer))

<u>Case – III:</u> **β < ω ⇒ Under Damped Harmonic Motion**

$$\beta^2 - \omega^2 < 0 \qquad \Rightarrow \left(\beta^2 - \omega^2\right)^{\frac{1}{2}} = \left\{-\left(\omega^2 - \beta^2\right)\right\}^{\frac{1}{2}}$$

$$\Rightarrow \left(\beta^2 - \omega^2\right)^{\frac{1}{2}} = \pm i\omega_1 \quad \left\{\text{Taking } \omega_1^2 = \left(\omega^2 - \beta^2\right)\right\}$$

So the solution can be written as

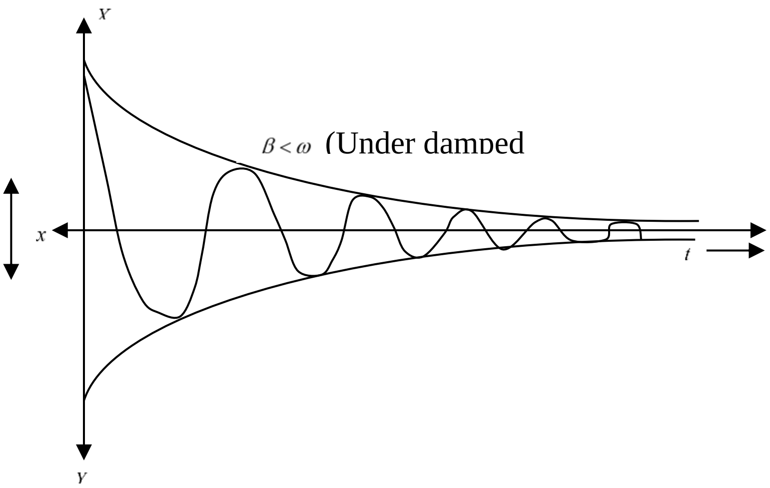
$$x = Ae^{-\beta t}\left[e^{(\beta^2-\omega^2)^{1/2}t} + e^{-(\beta^2-\omega^2)^{1/2}t}\right] \quad \text{taking, } A_1 = A_2 = A$$

$$x = Ae^{-\beta t}\left[e^{i\omega_1 t} + e^{-i\omega_1 t}\right]$$

$$x = Ae^{-\beta t}\left[\cos\omega_1 t + i\sin\omega_1 t + \cos\omega_1 t - i\sin\omega_1 t\right]$$

$$\boxed{x = 2A \cdot e^{-\beta t} \cdot \cos\omega_1 t} \qquad \text{---} \qquad (8)$$

The presence of ***cosω₁t*** term implies that the oscillator will oscillate and the presence of $e^{-\beta t}$ term in the amplitude implies that amplitude of oscillation decays exponentially. So, in under damped oscillatory motion the oscillator oscillates with reducing amplitude. So, the time period of oscillation of such oscillator is,



$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{(\omega^2 - \beta^2)^{1/2}}$$

And, time period of oscillation of undarmped oscillator is, $T = \frac{2\pi}{\omega}$

From the above two expressions it can be seen that $T_1 > T$

So, the time period of an under damped oscillator is greater than the undamped oscillator.

Hence the Frequency is, $f_1 = \frac{1}{T_1} = \frac{1}{2\pi}\left(\omega^2 - \beta^2\right)^{\frac{1}{2}}$ and $f_1 < f$

Energy of underdamped oscillator

Energy of underdamped oscillator can be written as

$$E = \frac{1}{2}kx^2$$

$$\Rightarrow E = \frac{1}{2}m\omega^2\left(2Ae^{-\beta t}\cos\omega_1 t\right)^2 \;\Rightarrow E = \frac{1}{2}m\omega^2 4A^2 e^{-2\beta t}\cos^2\omega_1 t \;\{\text{maximum value of } \cos^2\theta = +1\}$$

$$\Rightarrow E_{\max} = 2m\omega^2 A^2\left(e^{-2\beta t}\right) \quad \Rightarrow E_{\max} \propto e^{-2\beta t}$$

Since the oscillator oscillates with reducing amplitude, therefore the average energy is,

$$E_{av} = \left\langle \frac{1}{2}m\omega^2 4A^2 e^{-2\beta t}\cos^2\omega_1 t \right\rangle$$

Since the average value of $\cos^2\theta = \frac{1}{2}$ and other parameters are constants, therefore,

$$E_{av} = \frac{1}{2}m\omega^2 4A^2\left(e^{-2\beta t}\right)\frac{1}{2} \qquad \Rightarrow E_{av} = m\omega^2 A^2\left(e^{-2\beta t}\right)$$

Hence the energy of underdamped oscillator decreases with time and is proportional to $e^{-2\beta t}$.

Q Factor or Quality Factor of Oscillator

The Q factor of an under damped oscillator is defined as the number of oscillations through which the oscillator goes such that the energy decreases to $\frac{1}{e}$ of its maximum value.

We know, $E \propto e^{-2\beta t} \qquad \Rightarrow E = E_0 \cdot e^{-2\beta t}$

When, $t = \frac{1}{2\beta}$, $\boxed{E = \frac{E_0}{e}}$

During T_1 time, the oscillator covers 2π.

In 1 sec, the oscillator will cover $\frac{2\pi}{T_1}$.

In $\frac{1}{2\beta}$ sec, the oscillator will move $\frac{2\pi}{T_1}\times\frac{1}{2\beta} = \frac{\omega_1}{2\beta}$

So the Q factor is, $\boxed{Q = \frac{\omega_1}{2\beta}}$

Now, $E = E_0 \cdot e^{-2\beta t} \quad \Rightarrow dE = -2\beta E_0 \cdot e^{-2\beta t}dt$

Energy lost per cycle is,

$$-dE = 2\beta E_0 \cdot e^{-2\beta t}T_1 \quad \Rightarrow -dE = E_0 \cdot e^{-2\beta t}\times\frac{\omega_1}{Q}\times\frac{2\pi}{\omega_1}$$

$$\Rightarrow Q = \frac{2\pi E_0 \cdot e^{-2\beta t}}{-dE} \boxed{\Rightarrow Q = 2\pi\cdot\frac{\text{Energy stored per cycle}}{\text{Energy lost per cycle}}}$$

The quality factor can also be defined as 2π times the ratio of energy stored per cycle to the energy lost per cycle.

Decrement of the Oscillator :

For underdamped oscillator, we know,

$$x = 2Ae^{-\beta t}\cos\omega_1 t$$

Let A_0, A_1, A_2 ………… be the amplitudes at time t = 0, T_1, $2T_1$, ……………

So, at t = 0, $A_0 = 2A$

t = T_1, $A_1 = 2Ae^{-\beta T_1}$

t = $2T_1$, $A_1 = 2Ae^{-2\beta T_1}$

$\frac{A_0}{A_1} = \frac{A_1}{A_2} = ... = e^{\beta T_1}$ In general, $\boxed{\frac{A_{n-1}}{A_n} = e^{\beta T_1}}$ ------ Decrement of the oscillator.

The logarithm of this decrement is called the logarithmic decrement and it measures the rate of decrease of the amplitude of an under damped oscillator and denoted by λ.

So, $$\lambda = ln\frac{A_{n-1}}{A_n} = lne^{\beta T_1} = \beta T_1 = \frac{2\pi\beta}{\left(\omega^2 - \beta^2\right)^{\frac{1}{2}}}$$

$$\Rightarrow \lambda == \frac{2\beta T_1}{\left(\omega^2 - \beta^2\right)^{\frac{1}{2}}}$$

Modulus of Decay / Relaxation Time

It is defined as the time interval during which the amplitude decreases to $\frac{1}{e}$ of its original value.or maximum value.

The amplitude is given by $Ae^{-\beta t}$. When $t = \frac{1}{\beta}$, the Amplitude is, $\frac{A}{e}$

So, $\boxed{t = \frac{1}{\beta} = \tau = \text{Relaxation time}}$

4. FORCED HARMONIC MOTION (FHM):

It is that type of harmonic motion in which in addition to the restoring force and external resistive force, an external periodic force acts such that the oscillator continues to oscillate.

Equation of Motion

Consider a particle of mass 'm' executing FHM with displacement 'x'. The forces acting are,

(i) Force responsible for harmonic motion.

$F_1 = m\frac{d^2x}{dt^2}$ -------------- **(1)**

(ii) The restoring force proportional to the displacement and takes place in a direction opposite to it.

$F_2 = -kx$ --------------- **(2)**

k = proportionality constant known as spring constant.

(iii) External resistive force proportional to the velocity and takes place in a direction opposite to it

$F_3 = -R\frac{dx}{dt}$ ----------------- **(3)**

R = Proportionality constant and is called frictional force per unit velocity.

(iv) External periodic force, $F_4 = A_1 \cos \omega_1 t$ ………….. (4)

$\omega_1 =$ Angular frequency of oscillation of external periodic force

A_1 = Amplitude of external periodic force

Under equilibrium Condition,

$F_1 = F_2 + F_3 + F_4$

$$\Rightarrow \underline{m\frac{d^2x}{dt^2} = -kx - R\frac{dx}{dt} + A_1 cos\, \omega_1 t}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = A_1 cos\, \omega_1 t$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{R}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{A_1}{m} cos\, \omega_1 t$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = \frac{A_1}{m} cos\, \omega_1 t \qquad \text{----------} \qquad \textbf{(5)}$$

Equation (5) represents the equation of motion to a particle executing Forced Harmonic Motion.

Where, $\beta = \frac{R}{2m} =$ Damping co-efficient and $\omega = \sqrt{k/m} =$the restoring force.

Equation (5) is a non homogenous second order differential equation whose solution contains two parts that is, complementary function (x_c) and particular integral (x_p). So the solution can be written as,

$$x = x_C + x_P$$

$$\Rightarrow x = Ae^{-\beta t}\left[e^{t\sqrt{(\beta^2-\omega^2)}} + e^{-t\sqrt{(\beta^2-\omega^2)}}\right] + \frac{\left(A_1/m\right)}{\left[\left[\omega^2-\omega_1^2\right]^2+4\omega_1{}^2\beta^2\right]^{1/2}} cos(\omega_1 t - \delta) \quad \text{------} \quad \textbf{(6)}$$

As we are considering forced harmonic motion, the first term can be neglected because it is an exponentially decaying term due to $e^{-\beta t}$. So, the first part of the solution is called transient part and can be neglected.

The second part is called as steady state solution due to the presence of cosine term which indicates that the oscillator oscillates.

Now the solution can be written as

$$x = \frac{\left(A_1/_m\right)}{\left[\left[\omega^2-\omega_1^2\right]^2+4\omega_1{}^2\beta^2\right]^{1/2}} cos(\omega_1 t - \delta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (7)$$

$$\Rightarrow x = A\, cos(\omega_1 t - \delta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (8)$$

Equation **(7)** and **(8)** represent the displacement of a particle executing forced harmonic motion.

Where, $A = \frac{\left(A_1/_m\right)}{\left[\left[\omega^2-\omega_1^2\right]^2+4\omega_1{}^2\beta^2\right]^{1/2}}$ --------- (9) is called the amplitude of oscillation.

$\delta = tan^{-1}\left\{\frac{2\beta\omega_1}{(\omega^2-\omega_1{}^2)}\right\}$ -------------(10) is called the phase difference between the oscillator and external periodic force.

Velocity (v):

Velocity can be defined as the time rate of change of displacement, that is, $\mathrm{v} = \frac{dx}{dt}$

$$\Rightarrow \mathrm{v} = \frac{d}{dt}\left[\frac{A_1/_m}{\left[\left(\omega^2-\omega_1^2\right)+4\beta^2\omega_1^2\right]^{1/2}} cos(\omega_1 t - \delta)\right]$$

$$\Rightarrow \mathrm{v} = \left[\frac{-\left(A_1/_m\right)\omega_1}{\left[\left(\omega^2-\omega_1^2\right)+4\beta^2\omega_1^2\right]^{1/2}} sin(\omega_1 t - \delta)\right]$$

$$\Rightarrow \mathrm{v} = \left[\frac{\left(A_1/_m\right)\omega_1}{\left[\left(\omega^2-\omega_1^2\right)+4\beta^2\omega_1^2\right]^{1/2}} cos\left\{(\omega_1 t - \delta) + \frac{\pi}{2}\right\}\right] \text{--------}(11)\left\{\because cos\left(\theta + \frac{\pi}{2}\right) = -sin\,\theta\right\}$$

From the above equation, it has been observed that the velocity of a forced oscillator is leading the displacement by a phase of **π/2**.

Average Power Dissipated or Absorbed:

The amount of power dissipated or lost due to the damping force is compensated by the amount of power absorbed due to the external periodic force.

If we can obtain the amount of power dissipated, then, it will be the same as the amount of power absorbed.

So, the amount of power dissipated is,

$$P = \vec{F}_3.\vec{v}$$

$\Rightarrow P$ = (Rv) v cos 180^0 (As F_3&v are appositely directed)

$\Rightarrow P = -Rv^2$

$$\Rightarrow P = -R\left\{\frac{-\left(A_1/_m\right)\omega_1}{[(\omega^2-\omega_1{}^2)+4\beta^2\omega_1{}^2]^{1/2}}\sin(\omega_1 t-\delta)\right\}^2 \qquad \left(\because \beta = \frac{R}{2m}\right)$$

$$\Rightarrow P = 2m\beta A^2\omega_1{}^2\sin^2(\omega_1 t-\delta) \qquad \text{(Taking the magnitude)}$$

Since the power dissipated or absorbed oscillates between zero to maximum due to the presence of $sin^2(\omega_1 t-\delta)$ term, so the average power can be calculated by taking the average of the above equation.

$$P_{av} = \langle P\rangle = \langle 2m\beta A^2\omega_1{}^2\sin^2(\omega_1 t-\delta)\rangle$$

$\Rightarrow P_{av} = \langle 2m\beta A^2\omega_1{}^2\sin^2(\omega_1 t-\delta)\rangle$

$\Rightarrow P_{av} = 2m\beta A^2\omega_1{}^2\left(1/_2\right)$, ($\because \sin^2\theta$ oscillates between 0 to +1, so the average is $1/_2$)

$\Rightarrow P_{av} = m\beta A^2\omega_1{}^2$ ------------------------(12)

Total Average Energy (E):

The total energy of a particle executing forced oscillation is the sum of K.E & P.E

$\therefore E = E_K + E_P$ $\Rightarrow E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$$\Rightarrow E = \frac{1}{2}mA^2\omega_1{}^2\cos^2\left\{(\omega_1 t-\delta)+\pi/_2\right\} + \frac{1}{2}m\omega^2A^2\cos^2\omega\,(\omega_1 t-\delta) \quad \left(\because \omega = \sqrt{k/_m}\right)$$

$$E_{av} = \langle E\rangle = \left\langle\left[\frac{1}{2}mA^2\omega_1{}^2\cos^2\left\{(\omega_1 t-\delta)+\frac{\lambda}{2}\right\}\right] + \left[\frac{1}{2}m\omega^2A^2\cos^2(\omega_1 t-\delta)\right]\right\rangle$$

$$\Rightarrow E_{av} = \frac{1}{2}mA^2\omega_1{}^2\times\frac{1}{2} + \frac{1}{2}m\omega^2A^2\times\frac{1}{2} \qquad \left(\because \text{Average of } \cos^2\theta = \frac{1}{2}\right)$$

$\Rightarrow E_{av} = \frac{1}{4}mA^2(\omega_1{}^2+\omega^2)$ ……………………… (13)

The above equation represents the average energy of a forced harmonic oscillator.

Amplitude (A):

$$x = \frac{\left(A_1/_m\right)}{[(\omega^2-\omega_1{}^2)+4\beta^2\omega_1{}^2]^{1/2}} cos(\omega_1 t - \delta)$$

$$A = \frac{\left(A_1/_m\right)}{[(\omega^2-\omega_1{}^2)+4\beta^2\omega_1{}^2]^{1/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

The amplitude is maximum or minimum depending upon the term$(\omega^2 - \omega_1{}^2) + 4\beta^2\omega_1{}^2$. If this term is maximum then amplitude is minimum & vice versa.

5. RESONANCE:

The resonance can be defined as the maximum amplitude with which the oscillator oscillates when the time period of the oscillator and the time period of external periodic force become equal to each other.

OR

Resonance is said to occur when the oscillator and the external periodic force are in phase with each other and as a result the oscillator oscillates with maximum amplitude.

$$A = \frac{\left(A_1/_m\right)}{[(\omega^2-\omega_1{}^2)+4\beta^2\omega_1{}^2]^{1/2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

$(\omega^2 - \omega_1{}^2) + 4\beta^2\omega_1{}^2$ should be minimum.

$$\therefore \frac{d}{d\omega_1}[(\omega^2 - \omega_1{}^2)^2 + 4\beta^2\omega_1{}^2] = 0$$

$$\Rightarrow (-)2(\omega^2 - \omega_1{}^2) \times 2\omega_1 + 8\beta^2\omega_1 = 0$$

$$\Rightarrow -4\omega_1(\omega^2 - \omega_1{}^2) + 8\beta^2\omega_1 = 0]$$

$$\Rightarrow 4\omega_1(2\beta^2 - \omega^2 + \omega_1{}^2) = 0$$

$\Rightarrow \omega_1{}^2 - \omega^2 + 2\beta^2 = 0 \; as \;\; \omega_1 \neq 0$ being the frequency of external periodic force.

$$\Rightarrow \omega_1{}^2 = (\omega^2 - 2\beta^2) \Rightarrow \omega_1 = (\omega^2 - 2\beta)^{1/2} = \omega_R(say) \quad \ldots\ldots\ldots\ldots\ldots\ldots \; (16)$$

This equation represents the condition for resonance and ω_R is the value of ω_1 at resonance and the corresponding frequency is called as resonant frequency.

So, the oscillator oscillates with maximum amplitude at resonance when, $\omega_1 = (\omega^2 - 2\beta)^{1/2}$

Therefore, the maximum amplitude at resonance will be,

$$A_{max} = \frac{(A_1/m)}{\left[\left\{\omega^2 - (\omega^2 - 2\beta^2)\right\}^2 + 4\beta^2(\omega^2 - 2\beta^2)\right]^{1/2}}$$ (putting the condition for resonance, eqn - 16)

$$\Rightarrow A_{max} = \frac{(A_1/m)}{(4\beta^4 + 4\beta^2\omega^2 - 8\beta^4)^{1/2}}$$

$$\Rightarrow A_{max} = \frac{(A_1/m)}{(4\beta^2\omega^2 - 4\beta^4)^{1/2}} \quad \Rightarrow A_{max} = \frac{(A_1/m)}{\left[4\beta^2(\omega^2 - \beta^2)\right]^{1/2}}$$

$$\Rightarrow A_{max} = \frac{A_1}{2m\beta(\omega^2 - \beta^2)^{1/2}} \quad \text{……………………. (17)}$$

$$\Rightarrow A_{max} = \frac{A_1}{R(\omega^2 - \beta^2)^{1/2}} \quad \text{……………………... (18)}$$

For weak damping, $\omega >> \beta$, so, $A_{max} = \frac{A_1}{2m\beta\omega}$ or $\frac{A_1}{R\omega}$ ………………(19)

$$f_R = \frac{(\omega^2 - 2\beta^2)^{1/2}}{2\pi} \quad \text{………………. (20)}$$

The velocity of the oscillator at resonance is called as velocity resonance and can be written as,

$$\text{v} = A\omega_1 \cos\left\{(\omega_1 t - \delta) + \frac{\pi}{2}\right\}$$

$$\Rightarrow \text{v}_{max} = \frac{A_1}{2m\beta(\omega^2 - \beta^2)^{1/2}} \times (\omega^2 - 2\beta^2)^{1/2}$$ putting the value of A and maximum value of $\cos\theta$ is 1.

$$\Rightarrow \text{v}_{max} = \frac{A_1}{2m\beta} \times \left[\frac{(\omega^2 - 2\beta^2)}{(\omega^2 - \beta^2)}\right]^{1/2} \quad \text{……………………… (21)}$$

Equation (21) is called as velocity of oscillator at resonance or resonant velocity.

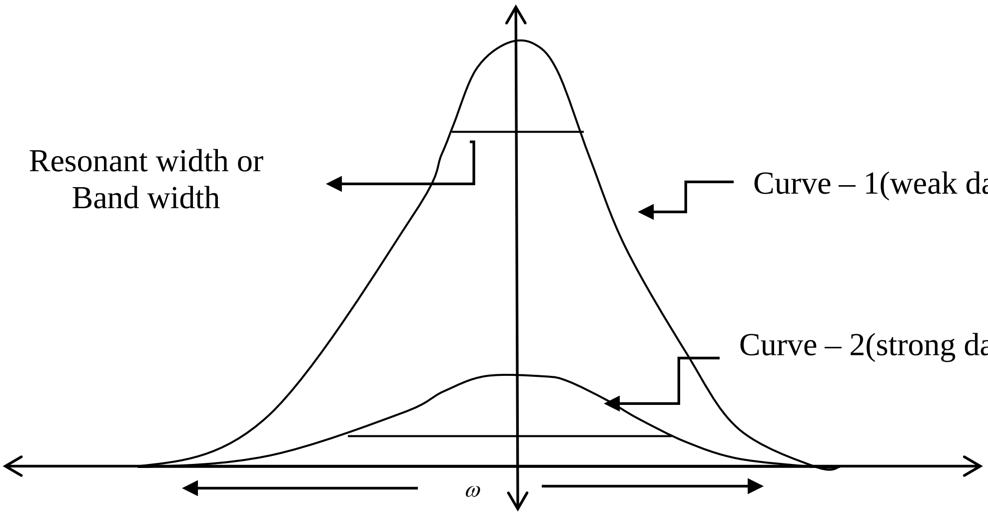
For weak damping, $\omega >> \beta$, so, $\text{v}_{max} \cong \frac{A_1}{2m\beta}$ or $\frac{A_1}{R}$ …………….. (22)

For weak damping, the velocity of oscillation is in phase with the external periodic force. Therefore, the external force acts always in the direction of the motion of the oscillator & hence the energy transfer is maximum. So the oscillator oscillates with maximum amplitude.

Sharpness of Resonance:

For, $\omega_1 = \omega \pm \beta$, the amplitude becomes, $A = \frac{A_{max}}{\sqrt{2}}$

Width of resonance can be defined as the width of the resonant curve at the point when the amplitude falls to $\frac{1}{\sqrt{2}}$ of its maximum value.



So, the width of resonance will be, $\Delta\omega_1 = (\omega + \beta) - (\omega - \beta) = 2\beta$

From the above equation, it is observed that,

a. When, β is small that is, for weak damping, the width of resonance decreases and hence the sharpness increases. (Curve 1)

b. When, β is more that is, for strong damping, the sharpness of resonance decreases and the curve becomes flat. (Curve 2)

Examples of Resonance:

SL. NO.	**NAME**	**DRIVEN**	**DRIVER**
1	Sonometer – Used for determining the frequency of unknown tuning fork.	Stretched string	Tuning Fork
2	Resonance column apparatus – Used for determining the velocity of sounds.	Air in the hollow tube	Tuning fork
3	Playing with a swing with maximum amplitude using external periodic force.	Swing	Force given by legs
4	Marching of military troops over a long suspension bridge.	Bridge	Force given by the legs of the troops

Advantages of Resonance:

1. Resonance is very often useful in finding an unknown frequency or in detecting a particular frequency present in a sound that consists of the mixture of frequency.
2. Resonance is also utilized in string instruments like violin, sitar etc where one additional string other than the main string is given to tune the desired modes of the scale.

Disadvantages of Resonance:

1. Sharp resonance in many cases is undesirable.
2. A microphone or loud speaker or any other instrument for recording or reproducing music must have the flat response to the range of frequency meant to reproduce. Resonance to any other frequency within the range will result in distortion.

Quality factor (Q):

The quality factor can be defined as $\frac{2\pi}{T}$ times the ratio of average energy stored per cycle to the average power dissipated per cycle at resonance.

Mathematically, $Q = \left[\left(\frac{2\pi}{T}\right)\frac{E_{av}}{P_{av}}\right]_{Resonance}$

$$\Rightarrow Q = \left\{\frac{2\pi}{T} \times \frac{\frac{1}{4}mA^2\left(\omega^2 + \omega_1^{\ 2}\right)}{m\beta A^2 \omega_1^{\ 2}}\right\}_{Resonance}$$

$$\Rightarrow Q = \left\{\frac{2\pi}{T} \times \frac{\frac{1}{4}mA^2\omega_R\left(\omega^2 + \omega_R^{\ 2}\right)}{m\beta A^2 \omega_R^{\ 2}}\right\} \quad \left[\omega_1 = \omega_R \text{ at resonance and } \frac{2\pi}{T} = \omega_R\right]$$

$$\Rightarrow Q = \frac{\left(\omega^2 + \omega_R^{\ 2}\right)}{4\beta\omega_R} \qquad \Rightarrow Q = \frac{2\omega_R^{\ 2}}{4\beta\omega_R} \quad \left(\because \omega_R \cong \omega \text{ for weak damping}\right)$$

$$\Rightarrow Q = \frac{\omega_R}{2\beta} \qquad \Rightarrow Q = \frac{\text{Resonant Frequency}}{\text{Width of Resonance}} \quad \ldots\ldots\ldots\ldots\ldots(23)$$

So, the quality factor can also be defined as the ratio of resonant frequency to width of resonance.

The quality factor of different oscillators is given below.

Name of The Oscillator	Value of Q-Factor	Name of The Oscillator	Value of Q-Factor
Earth quake	250 - 1400	Micro wave resonant	10^5
Violin string	10^3	Crystal oscillator	10^6
Excited atom		10^8	

6. Coupled Oscillation:

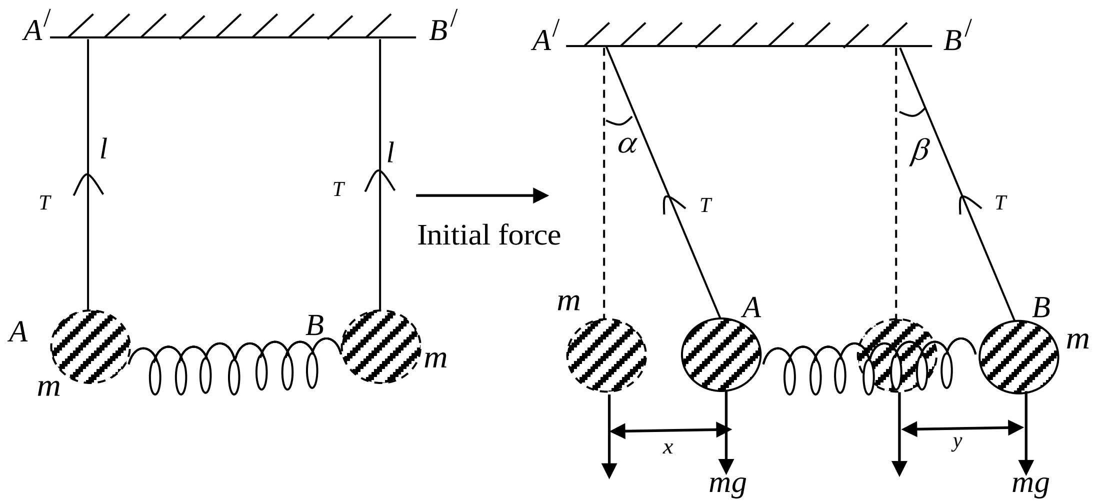
If two or more oscillators are connected in such a way that there is transfer of energy among the oscillators, then the oscillation of the system of oscillators is called **coupled oscillation**. The oscillator is called as coupled oscillator.

Examples:

1. Two or more pendulums connected by means of a light spring suspended from one rigid support.
2. The prongs of tuning fork.
3. Atoms or ions or molecules in a solid.

Theory of Oscillation of two Oscillators Coupled with each other:

Consider two oscillators A & B of mass 'm' suspended from support A′B′ by means of weightless strings of length 'l'. The two oscillators are connected with each other by means of a spring. Now the two oscillators A & B are displaced to a distance x & y such that both of them oscillates. T is the tension on the string acting vertically upward. mg is the weight of the body acting downward. α and β are the angles made by the oscillators A and B with the verticals.



1. **Equation of motion for A:**

   a. Force responsible for motion, $F_1 = m\frac{d^2x}{dt^2}$

   b. Restoring force due to the spring on A, $F_2 = -k\,(x - y)$

   c. Restoring force on A due to the component of the weight of the body,

   $$F_3 = -mg\sin\alpha = -mg.(x/l)$$

Under equilibrium condition,

$$F_1 = F_2 + F_3$$

$$\Rightarrow F_1 - F_2 - F_3 = 0$$

$$\Rightarrow \frac{m\cdot d^2x}{dt^2}+k\left(x-y\right)+mg\cdot\frac{x}{l}=0$$

$$\Rightarrow \frac{m\cdot d^2x}{dt^2}+\frac{k}{m}\left(x-y\right)+\frac{mg\cdot\frac{x}{l}}{m}=0$$

$$\Rightarrow \frac{d^2x}{dt^2}+\omega^2\left(x-y\right)+\frac{x\cdot g}{l}$$

$$\Rightarrow \frac{d^2x}{dt^2}+\omega^2\left(x-y\right)+\omega_1 x=0 \qquad \text{---------------} \quad (1) \qquad \text{where } \omega_1{}^2=\frac{g}{l}$$

2. **Equation of motion for B:**

   a. Force responsible for motion, $F_1 = m\frac{d^2y}{dt^2}$

   b. Restoring force due to the spring on A, F_2 =-k (y - x)

   c. Restoring force on A due to the component of the weight of the body,

$$F_3 = -mg\sin\beta = -mg.(^y/_l)$$

Under equilibrium condition,

$$F_1=F_2+F_3$$

$$F_1-F_2-F_3=0$$

$$\Rightarrow \frac{m\cdot d^2y}{dt^2}+k\left(y-x\right)+mg\cdot\frac{y}{l}=0$$

$$\Rightarrow \frac{\frac{m\cdot d^2y}{dt^2}}{m}+\frac{k}{m}\left(y-x\right)+\frac{mg}{m}\frac{y}{l}=0$$

$$\Rightarrow \frac{d^2y}{dt^2}+\frac{k}{m}\left(y-x\right)+\frac{g}{l}y=0$$

$$\Rightarrow \frac{d^2y}{dt^2}-\frac{k}{m}\left(x-y\right)+\omega_1{}^2y=0 \qquad \text{---------------} \quad (2) \qquad \text{where } \omega_1{}^2=\frac{g}{l}$$

Now, adding equations (1) & (2), we get

$$\frac{d^2}{dt^2}(x+y)+\omega_1{}^2(x+y)=0 \qquad \text{------} \ (3)$$

Subtracting equation (2) from (1), we get

$$\frac{d^2}{dt^2}(x-y)+\omega_1{}^2(x-y)+\frac{2k}{m}(x-y)$$

$$\Rightarrow \qquad \frac{d^2}{dt^2}(x-y)+\left(\omega_1{}^2+\frac{2k}{m}\right)(x-y)=0$$

$$\Rightarrow \qquad \frac{d^2}{dt^2}(x-y)+\omega_2{}^2(x-y)=0 \qquad \text{---------}(4)$$

where, $\omega_2=\sqrt{\left(\frac{2k}{m}+\frac{g}{l}\right)}$

Equations (3) & (4) represent a set of coupled equality in x and y. Such equations can be solved by using normal co-ordinates.

Normal co-ordinates are the coordinates which are linear combination of the variables taken. Such coordinates are used to decouple the set of coupled equations.

The solutions obtained in times of normal coordinates represent normal modes of oscillation and the frequencies with which the oscillator oscillates are called normal mode frequencies.

Let the normal coordinates be, Q_1 and Q_2, such that,

$$Q_1 = x + y \ \ \& \ \ Q_2 = x - y$$

So, equation (3) $\dfrac{d^2Q_1}{dt^2} + \omega_1^2 Q_1 = 0$ ……………………… (5)

Equation (5) $\dfrac{d^2Q_2}{dt^2} + \omega_2^2 Q_2 = 0$ ……………………... (6)

Equation (5) & (6) represent equations of motion of coupled oscillators decoupled using normal coordinates.

The solution of the set of decoupled equations implies that the coupled oscillator will oscillate in Q_1 mode with frequency

$$\omega_1 = \sqrt{\frac{g}{l}} \quad \text{and} \quad f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad \text{………………} \quad (7)$$

or in Q_2 mode with frequency

$$\omega_2 = \sqrt{\frac{2k}{m} + \frac{g}{l}} \quad \text{and} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{2k}{m} + \frac{g}{l}} \qquad \text{…………..} \quad (8)$$

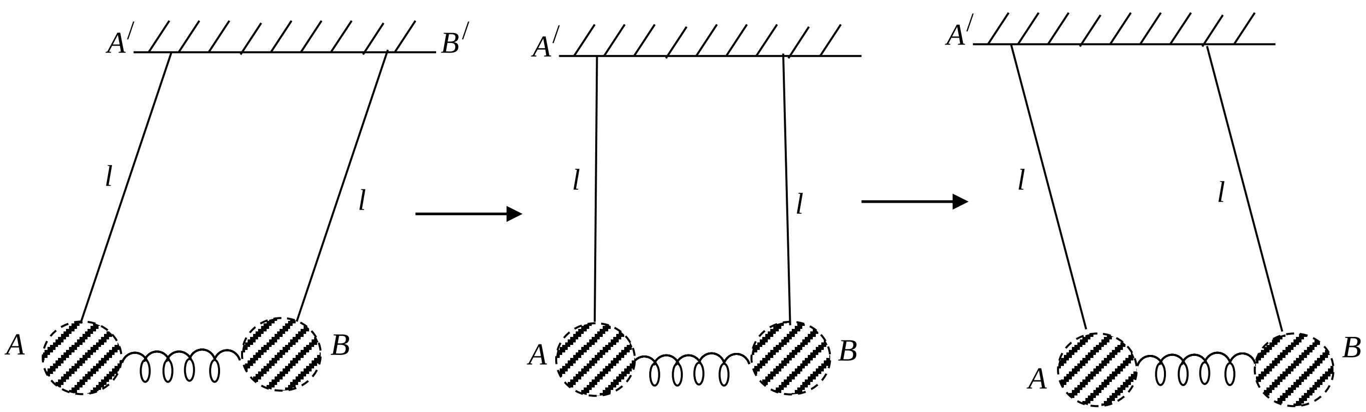
Frequencies represented in equations (7) & (8) are called normal mode frequencies.

Case – I Q_1 mode or in Phase Mode:

Let $x = y$

$Q_1 = x + y = 2x$ *or* $2y$ and $Q_2 = 0$

So, Q_1 mode of oscillation is active with frequency $f_1 = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}}$



Since, the two oscillators displaced same distance along same direction, so they are in phase with each other. Hence, the oscillation is called as Q_1 mode or in phase mode of oscillation. Here, the length of the spring does not change.

Case – II Q₂ mode or out of Phase Mode of oscillation:

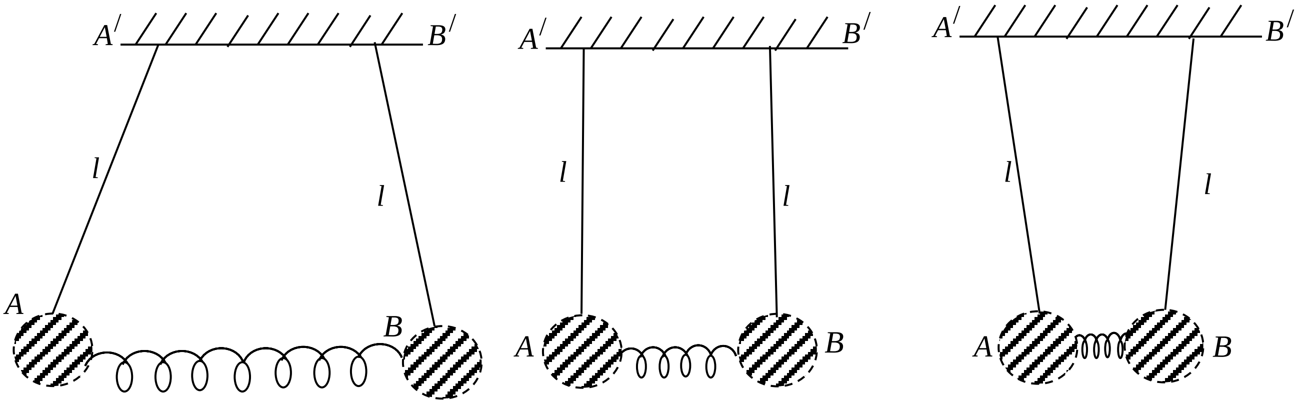
Let $x = -y$

$Q_1 = 0$

$Q_2 = finite$

Q_2 mode of oscillation is active with frequency $f_2 = \frac{1}{2\pi}\sqrt{\frac{2k}{m} + \frac{g}{l}}$

Since, the two oscillators displaced same distance along opposite direction, so they are out of phase with each other. Hence, the oscillation is called as Q_2 mode or out of phase mode of oscillation. Here the length of the spring changes that is, it expands and contracts.



7. Wave Motion

Spatial variation – Variation of any quantity with respect to Position.

Temporal variation – Variation of any quantity with respect to time.

Wave

It is the disturbance that travels from one position to the other in the medium due the periodic motion of the particles about their mean position.

Or

It is the disturbance that travels in a medium due to transfer of energy from particle to particle.

Characteristics

a. For a wave to propagate the medium should be elastic in nature.

b. Only the wave travels forward but the particles vibrate about their mean position.

c. The velocity with which the wave travels is called as waver velocity and the velocity with which the particles vibrate is called as particle velocity. Both are different from each other. The wave travels with uniform velocity whereas the particle velocity is not uniform.

d. When the wave travels from one medium to other, reflection and transmission takes place at the boundary separating the two media.

e. The velocity of the wave changes when it travels from one medium to the other medium.

i. It decreases when moves from rarer to denser medium.

ii. It increases when travels from denser to rarer medium.

f. Different physical parameters are used to represent a wave. They are amplitude, frequency, time period, angular frequency, wave velocity, wavelength, phase, path etc…

g. In order to describe a wave physically, a mathematical term is used which is called as wave function and is a function of position (x, y, z) and time (t). it is represented by $\psi(x, y, z, t)$.

In 1-D: $\psi(x,t)$ or $\psi(y,t)$ or $\psi(z,t)$

In 2-D: $\psi(x,\mathrm{y},t)$ or $\psi(y,\mathrm{z},t)$ or $\psi(z,x,t)$

In 3-D: $\psi(x,\mathrm{y},\mathrm{z},t)$

In general, the wave function is written as, $\psi(\vec{r},t)$, where, $\vec{r}$ – Position and t – Time

Examples of different types of wave: -

1. Directly described wave.

a. Waves on water surface.

b. Waves on strings.

c. Displacement of particles about their equilibrium position.

2. Indirectly described waves.

a. Propagation of electromagnetic waves.

b. Electric & magnetic waves in a circuit.

c. Sound waves

Different forms of wave function:

In a wave motion, the particles of the medium executes S.H.M.

So, the displacement is given by,

$$y = A\sin(\omega t + \phi) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad \mathbf{(1)}$$

where, ϕ = phase difference.

In terms of path difference we can write,

$$\text{Phase difference} = \frac{2\lambda}{\pi} \times \text{ Path difference.}$$

$$\Rightarrow \phi = \frac{2\pi}{\lambda} x.$$

So, the expression for wave function can written as, $\psi(x,t) = A\sin\left(\omega t + \frac{2\pi}{\lambda} x\right)$

For a wave, following substitutions can be made.

$$k = \frac{2\pi}{\lambda}, \qquad \omega = 2\pi f \qquad \mathrm{v} = f\lambda \qquad \text{and} \qquad f = \frac{1}{T}$$

where $k \rightarrow$ Propagation constant

$\lambda \rightarrow$ Wavelength

$\omega \rightarrow$ Angular frequency

$f \rightarrow$ Frequency

$T \rightarrow$ Time period

$\mathrm{v} \rightarrow$ Wave velocity

$x \rightarrow$ Path different between the two waves

$A \rightarrow$ Amplitude of wave

$t \rightarrow$ time for which the wave travels.

$$\psi(x,t) = A\sin(\omega t + kx) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \qquad \textbf{(2)}$$

$$\psi(x,t) = A\sin\left(2\pi ft + \frac{2\pi}{\lambda}x\right) \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad \textbf{(3)}$$

$$\psi(x,t) = A\sin\left(2\pi\frac{v}{\lambda}t + \frac{2\pi}{\lambda}x\right) \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad \textbf{(4)}$$

$$\psi(x,t) = A\sin\left\{\frac{2\pi}{\lambda}(vt + x)\right\} \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad \textbf{(5)}$$

$$\psi(x,t) = A\sin\left\{2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right)\right\} \ldots\ldots\ldots\ldots\ldots\ldots.. \qquad \textbf{(6)}$$

$$\vec{k} = \hat{k}\left|\vec{k}\right| = \text{ propagation vector}$$

$$= \hat{i}kx + \hat{j}ky + \hat{k}kz$$

In general, equation (3) can be written as,

$$\psi(x,\mathrm{y},\mathrm{z},t) = \psi(\vec{r},t) = A\sin\left\{\omega t \pm \left(k_x x + k_y y + k_z z\right)\right\}$$

$$\Rightarrow \psi(\vec{r},t) = A\sin\left(\omega t \pm \vec{k}\cdot\vec{r}\right)$$

$$\text{Since,} \quad \vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z = kx + ky + kz$$

Equations (2), (3), (4), (5) & (6) represent different forms of wave function

Other forms of wave function in different dimensions:

For 1 – D (Along the axis)		
Along X-axis $\psi = A\sin(\omega t \pm kx)$ $\psi = A\cos(\omega t \pm kx)$ $\psi = Ae^{\pm i(\omega t \pm kx)}$	**Along Y-axis** $\psi = A\sin(\omega t \pm ky)$ $\psi = A\cos(\omega t \pm ky)$ $\psi = Ae^{\pm i(\omega t \pm ky)}$	**Along Z-axis** $\psi = A\sin(\omega t \pm ky)$ $\psi = A\cos(\omega t \pm ky)$ $\psi = Ae^{\pm i(\omega t \pm ky)}$
For 2 – D (Plane)		
Along XY- plane $\psi = A\sin(\omega t \pm (kx + ky))$ $\psi = A\cos(\omega t \pm (kx + ky))$ $\psi = Ae^{\pm i(\omega t \pm (kx+ky))}$	**Along YZ- plane** $\psi = A\sin(\omega t \pm (ky + kz))$ $\psi = A\cos(\omega t \pm (ky + kz))$ $\psi = Ae^{\pm i(\omega t \pm (ky+kz))}$	**Along ZX- plane** $\psi = A\sin(\omega t \pm (kz + kx))$ $\psi = A\cos(\omega t \pm (kz + kx))$ $\psi = Ae^{\pm i(\omega t \pm (kz+kx))}$
For 3 – D (Space)		
$\psi = A\sin(\omega t \pm (kx + ky + kz)) = A\sin(\omega t \pm \vec{k}\cdot\vec{r})$ $\psi = A\cos(\omega t \pm (kx + ky + kz)) = A\cos(\omega t \pm \vec{k}\cdot\vec{r})$ $\psi = Ae^{\pm i(\omega t \pm (kx+ky+kz))} = Ae^{\pm i(\omega t \pm (\vec{k}\cdot\vec{r}))}$		

Differential Equation for a wave or Wave Equation:

Wave equation can be defined as a second order partial differential equation in space and time.

We know that, the expression for wave function is,

$$\psi = A\sin\left\{\frac{2\pi}{\lambda}(\mathrm{v}t + x)\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

Differentiating the above equation partially with respect to *x*, we can write,

$$\frac{\partial \psi}{\partial x} = A\frac{2\pi}{\lambda}\cos\left\{\frac{2\pi}{\lambda}(\mathrm{v}t + x)\right\}$$

Again, differentiating with respect to *x*,

$$\frac{\partial^2 \psi}{\partial x^2} = -A\frac{4\pi^2}{\lambda^2}\sin\left\{\frac{2\pi}{\lambda}(\mathrm{v}t + x)\right\} \quad \Rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{\lambda^2}\psi \qquad \ldots\ldots\ldots\ldots\ldots.. \quad (2)$$

Differentiating equation (1) partially with respect to t, we can write,

$$\frac{\partial \psi}{\partial t} = A\frac{2\pi}{\lambda}\mathrm{v}\cos\frac{2\pi}{\lambda}(\mathrm{v}t + x)$$ Again differentiating with respect to t,

$$\frac{\partial^2 \psi}{\partial t^2} = -A\frac{4\pi^2}{\lambda^2}\mathrm{v}^2\sin\left\{\frac{2\pi}{\lambda}(\mathrm{v}t + x)\right\} \quad \Rightarrow \frac{\partial^2 \psi}{\partial t^2} = -\frac{4\pi^2}{\lambda^2}\mathrm{v}^2\psi \qquad \ldots\ldots\ldots\ldots\ldots.. \quad (3)$$

Comparing equations (2) & (3) we can write,

$$\frac{\partial^2\psi}{\partial t^2} = \mathrm{v}^2 \frac{\partial^2\psi}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 y}{\partial t^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4)$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (5)$$

Equation (4) and (5) represent the one dimensional wave equation or differential equation for the wave function.

The wave equation for different dimensions can be written as,

For 1 – D (Along the axis)		
Along X-axis $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\frac{\partial^2\psi}{\partial x^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$	**Along Y-axis** $\frac{\partial^2\psi}{\partial y^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\frac{\partial^2\psi}{\partial y^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$	**Along Z-axis** $\frac{\partial^2\psi}{\partial z^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\frac{\partial^2\psi}{\partial z^2} - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$
For 2 – D (Plane)		
Along XY- plane $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$	**Along YZ- plane** $\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\left(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$	**Along ZX- plane** $\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial x^2}\right) - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$
For 3 – D (Space)		
$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ $\qquad \Rightarrow \nabla^2\psi = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2}$ Or $\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$ $\qquad \Rightarrow \nabla^2\psi - \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial t^2} = 0$ $where, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$		

Types of progressive waves

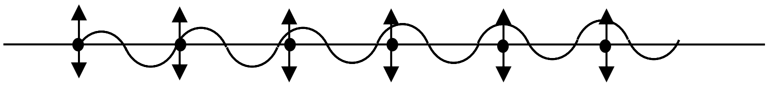
Depending upon the angle between the direction of vibration of particles and direction of propagation of waves, there are two types of progressive waves.

i. Transverse progressive wave.

ii. Longitudinal progressive wave.

i. Transverse progressive wave.

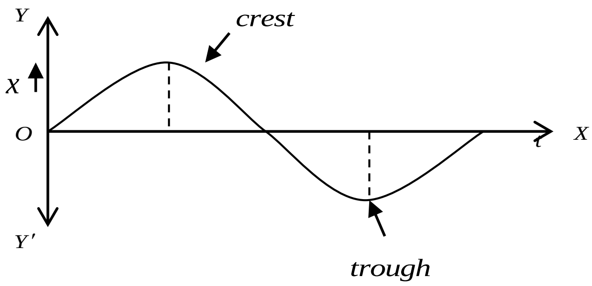
It is the type of wave motion in which the particle of medium vibrates about their mean position in a direction perpendicular to the direction of propagation of the wave.



Properties of transverse progressive wave

a. For transverse wave, the medium should possess the property of cohesion & elasticity. Hence transverse waves propagate in solids and cannot be propagated in gases as they do not possess cohesion.

b. Transverse waves travel in the form of crests and troughs. Crest is the position of maximum displacement along positive direction and trough is the position of maximum displacement along negative direction.



c. The vibrations in a medium are maintained due to the properties of elasticity and density. In case of a solid the elasticity is closely related to tension (T) and density is related to mass per unit length (m).

So, the speed of the transverse progressive wave is given by,

$$v_t = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{T}{m}}$$ where E is the elasticity and ρ is the density.

Examples:

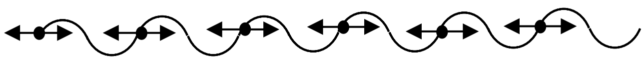
(1) Propagation of waves through solids.

(2) Vibration of strings fixed at one end.

(3) Electromagnetic waves.

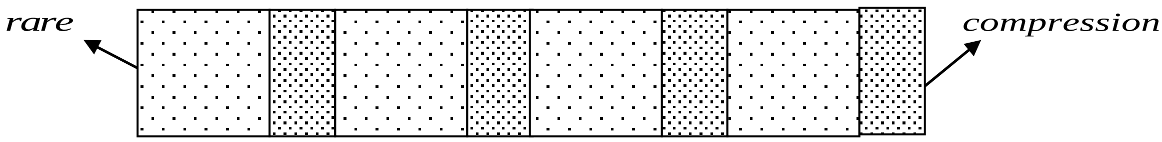
Longitudinal progressive wave:-

It is the type of wave motion in which the particles of the medium vibrate along the direction of propagation of the wave.



Properties Longitudinal progressive wave:-

a. Longitudinal waves travel in the medium in the form of compression and rare fractions. Compression is the region of maximum density and rare fraction is the region of minimum density.



b. The compressions and rare factions are maintained due to the elasticity and the density of the medium. Therefore, the velocity of the longitudinal wave is given by,

$$v_l = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$$

where, P = pressure, $\rho =$ density, $\gamma = \frac{C_p}{C_v}$ = adiabatic index,

C_p and C_v = specific heat at constant pressure and volume respectively.

c. There is always transfer of energy along the direction of propagation of the wave.

Examples:

(1) Elastic waves in liquid and gaseous medium.

(2) Propagation of sound waves.

(3) Waves on water surface is an example where both longitudinal and transverse waves exist.

Reflection & transmission of waves at the boundary separating two media

When a wave traveling through a given medium meets another medium, then a part of the incident wave is reflected and the rest part is transmitted to the other medium.

The characteristics of the reflected and transmitted waves are different from that of incident wave.

The change in characteristics are,

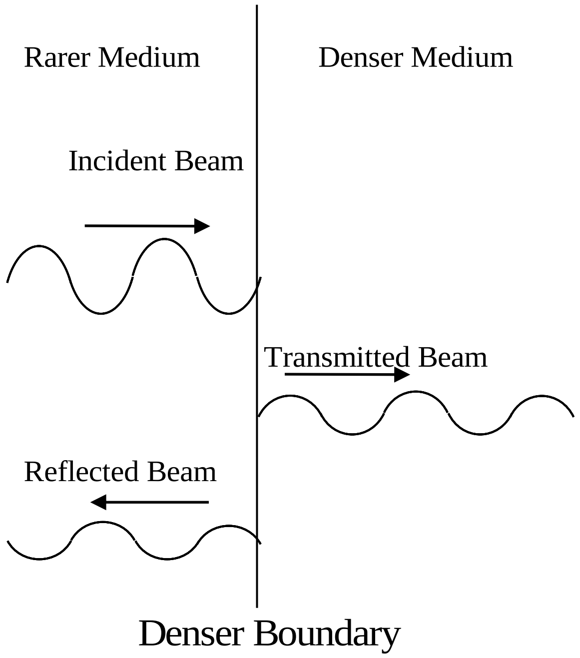
a. The amplitude of both reflected and transmuted waves are less than that of incident wave.

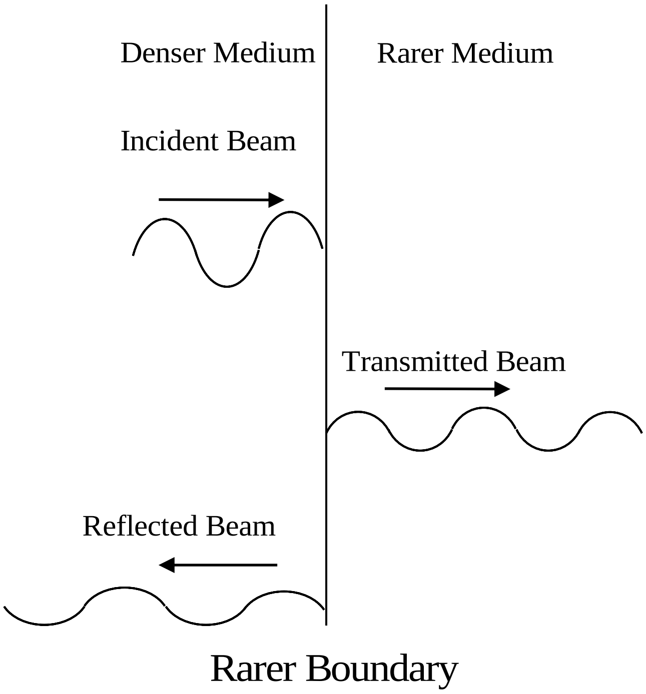
b. The frequency of reflected & transmuted waves is same as that of the incident wave.

c. The wave length of the reflected wave is same as that of the incident wave but in case of transmitted wave it changes.

d. The velocity of the reflected wave is same as that of the incident wave as they are in same medium. But in case of transmitted wave it changes. It decreases when the wave travel from rarer to denser medium and increases when it travels from denser to rarer medium.

e. There is no change in phase of the transmitted wave and reflected wave at the rarer boundary. The only change of phase occurs when the reflected wave meets the denser boundary. This phase change is π.





Reflectance or co-efficient of Reflection

It is defined as the ratio of intensity of reflected wave to the intensity of incident wave. Mathematically,

$$\text{Reflectance}(R) = \frac{\text{intensity of reflected wave}}{\text{intensity of incident wave}} = \frac{I_r}{I_i}$$

Transmittance or co-efficient of Transmission

It is defined as the ratio of intensity of transmitted wave to the intensity of incident wave. Mathematically,

$$\text{Transmittance }(T) = \frac{\text{intensity of transmitted wave}}{\text{intensity of incident wave}} = \frac{I_t}{I_i}$$

The intensity of incident wave is the sum of reflected wave & transmitted wave

So, $I_i = I_r + I_t \qquad \Rightarrow \frac{I_i}{I_i} = \frac{I_r}{I_i} + \frac{I_t}{I_i} \qquad \Rightarrow R + T = 1$